However, the question of whether or not an impulse in the spectrum represents a *visible* periodic component in the image strongly depends on properties of the human visual system. The fact that the eye cannot distinguish fine details above a certain frequency (i.e., below a certain period) suggests that the human visual system model includes a low-pass filtering stage. This is a bidimensional bell-shaped filter whose form is anisotropic (since it appears that the eye is less sensitive to small details in diagonal directions such as 45°).[7] However, for the sake of simplicity this low-pass filter can be approximated by the *visibility circle*, a circular step function around the spectrum origin whose radius represents the *cutoff frequency* (i.e., the threshold frequency beyond which fine detail is no longer detected by the eye). Obviously, its radius depends on several factors such as the contrast of the observed details, the viewing distance, light conditions, etc. If the frequencies of the original image elements are beyond the border of the visibility circle in the spectrum, the eye can no longer see them; but if a strong enough impulse in the spectrum of the image superposition falls inside the visibility circle, then a moiré effect becomes visible in the superposed image. (In fact, the visibility circle has a hole in its center, since very low frequencies cannot be seen, either.)

For the sake of convenience, we may assume that the given images (gratings, grids, etc.) are symmetrically centered about the origin. As a result, we will normally deal with images (and image superpositions) which are *real* and *symmetric*, and whose spectra are, consequently, also real and symmetric (Ref. 6, pp. 14, 15). This means that each impulse in the spectrum (except for the dc impulse at the origin) is always accompanied by a twin impulse of an identical amplitude, which is symmetrically located at the other side of the origin as in Fig. 2 (their frequency vectors are $\mathbf{f}$ and $-\mathbf{f}$). If the image is nonsymmetric (but, of course, still real), the amplitudes of the twin impulses at $\mathbf{f}$ and $-\mathbf{f}$ are complex conjugates.

2.2 *Spectrum Convolution and Superposition Moirés*

According to the convolution theorem [Eqs. (1), (2)], when m line gratings are superposed in the image domain, the resulting spectrum is the convolution of their individual spectra. This convolution of combs can be seen as an operation in which frequency vectors from the individual spectra are added vectorially, while the corresponding impulse amplitudes are multiplied. More precisely, each impulse in the spectrum convolution is generated during the convolution process by the contribution of *one* impulse from *each* individual spectrum: its location is given by the sum of their frequency vectors, and its amplitude is given by the product of their amplitudes. This permits us to introduce an indexing method for denoting each of the impulses of the spectrum convolution in a unique, unambiguous way. The general impulse in the spectrum convolution will be denoted the $(k_1,k_2,\dots,k_m)$ *impulse*, where m is the number of superposed gratings, and each integer k_i is the index (harmonic), within the comb (the Fourier series) of the ith spectrum, of the impulse that this ith spectrum contributed to the impulse in question in the convolution. Using this formal notation we can, therefore, express the geometric location of the general $(k_1,k_2,\dots,k_m)$ impulse in the spectrum convolution by the vectorial sum (or linear combination)

$$\mathbf{f}_{k_1,k_2,\dots,k_m}=k_1\mathbf{f}_1+k_2\mathbf{f}_2+\dots+k_m\mathbf{f}_m \tag{3}$$

and its amplitude by

$$a_{k_1,k_2,\dots,k_m}=a^{(1)}{}_{k_1}a^{(2)}{}_{k_2}\dots a^{(m)}{}_{k_m}, \tag{4}$$

where $\mathbf{f}_i$ denotes the frequency vector of the fundamental impulse in the spectrum of the ith grating, and $k_i\mathbf{f}_i$ and $a^{(i)}{}_{k_i}$ are, respectively, the frequency vector and the amplitude of the k_ith harmonic impulse in the spectrum of the ith grating.

The vectorial sum of Eq. (3) can also be written in terms of its Cartesian components. If f_i are the frequencies of the m original gratings and θ_i are the angles that they form with the positive horizontal axis, then the coordinates (f_u,f_v) of the $(k_1,k_2,\dots,k_m)$ impulse in the spectrum convolution are given by

$$\begin{aligned} f_{u_{k_1,k_2,\dots k_m}}&=k_1f_1\cos\theta_1+k_2f_2\cos\theta_2+\dots+k_mf_m\cos\theta_m\,,\\ f_{v_{k_1,k_2,\dots,k_m}}&=k_1f_1\sin\theta_1+k_2f_2\sin\theta_2+\dots+k_mf_m\sin\theta_m\,. \end{aligned} \tag{5}$$

Therefore, the frequency, the period, and the angle of the considered impulse (and of the moiré it represents) are given by the length and the direction of the vector $\mathbf{f}_{k_1,k_2,\dots,k_m}$ as follows:

$$f=\sqrt{f_u^2+f_v^2}\qquad T_M=1/f\qquad \varphi_M=\arctan(f_v/f_u). \tag{6}$$

Let us now say a word about the notations used for the superposition moirés. We use a notational formulation which provides a systematic means for identifying the various moiré effects. As we have seen, a $(k_1,k_2,\dots,k_m)$ impulse of the spectrum convolution which falls close to the spectrum origin, inside the visibility circle, represents a moiré effect in the superposed image (see Fig. 3). We call the m-grating moiré whose fundamental impulse is the $(k_1,k_2,\dots,k_m)$ impulse in the spectrum convolution a $(k_1,k_2,\dots,k_m)$ *moiré*; the highest absolute value in the index list is called the *order* of the moiré. Note that in the case of doubly periodic images, such as in dot screens, each image can be represented in the superposition by a pair of onefold periodic functions; hence, m in Eqs. (3)–(5) above counts each doubly periodic layer as two onefold periodic structures.

2.3 *Singular States; Stable Versus Unstable Moiré-Free Superpositions*

We have seen that if one or several of the new impulse pairs in the spectrum convolution fall close to the origin, inside the visibility circle, this implies the existence in the superposed image of one or several moirés with visible periods [see, for example, Figs. 3(c) and 3(f)]. An interesting special case occurs when some of the impulses of the convolution fall *exactly* on top of the dc impulse, at the

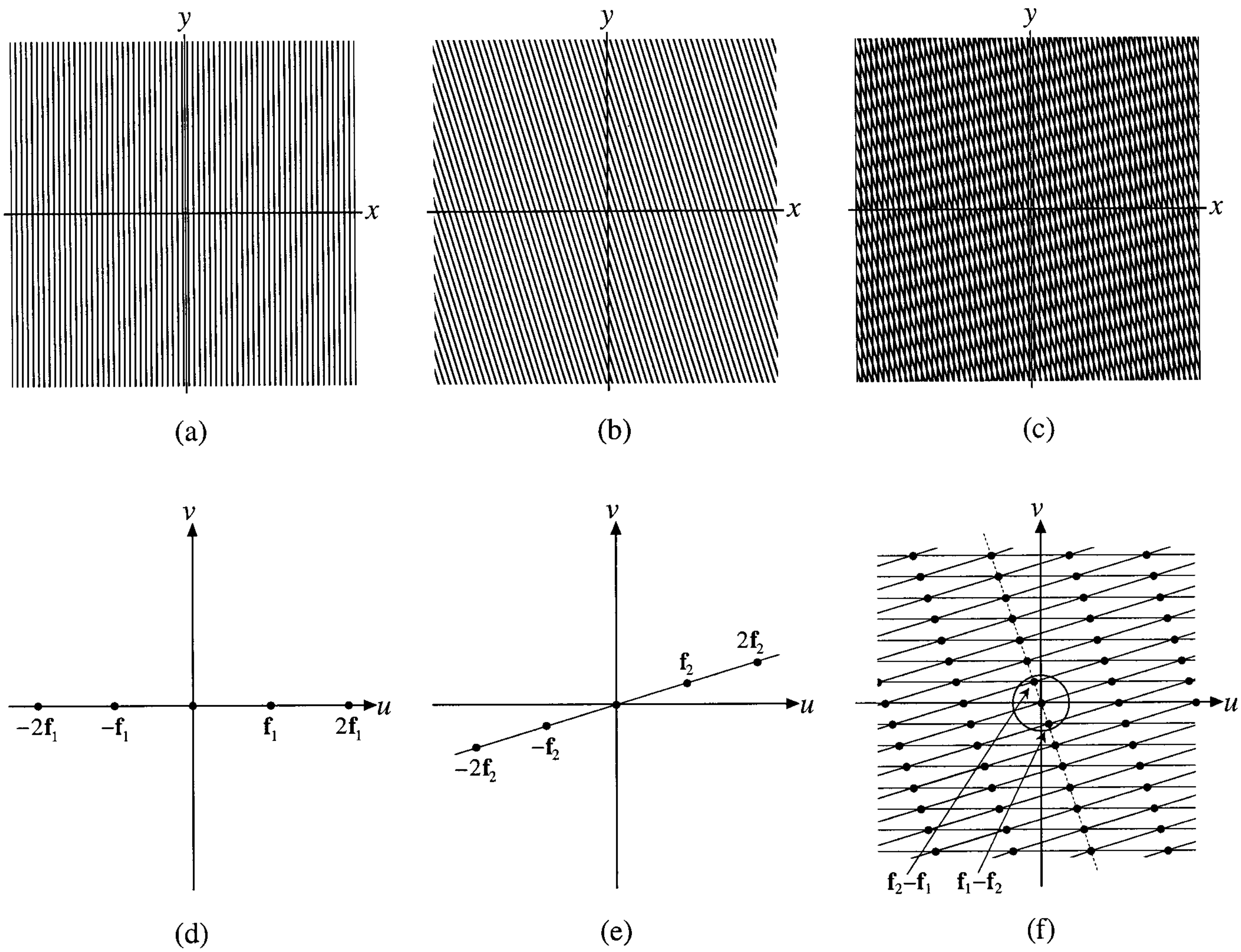

Fig. 3 Line gratings (a) and (b) and their superposition (c) in the image domain; their respective spectra are the infinite impulse combs shown in (d) and (e) and their convolution (f). Only impulse locations are shown in the spectra, but not their amplitudes. The circle in the center of the spectrum (f) represents the visibility circle. It contains the impulse pair whose frequency vectors are $\mathbf{f}_1-\mathbf{f}_2$ and $\mathbf{f}_2-\mathbf{f}_1$ and whose indices are $(1,-1)$ and $(-1,1)$; this is the fundamental impulse pair of the $(1,-1)$ moiré seen in (c). The dotted line in (f) shows the infinite impulse comb which represents this moiré.

spectrum origin. This happens, for instance, in the trivial superposition of two identical gratings in match, with an angle difference of 0° or 180°; or, more interestingly, when three identical gratings are superposed with angle differences of 120° between each other (see second and third rows of Fig. 4). As can be seen from the vector diagrams, these are limit cases in which the vectorial sum of the frequency vectors is exactly $\mathbf{0}$. This means that the moiré frequency is 0 (i.e., its period is infinitely large), and, therefore, as shown in Figs. 4(d) and 4(g), the moiré is not visible. This situation is called a *singular moiré state*. But, although the moiré effect in a singular state is not visible, this is a very unstable moiré-free state, since any slight deviation in the angle or in the frequency of any of the superposed layers may cause the new impulses in the spectrum convolution to move slightly off the origin, thus causing the moiré to "come back from infinity" and to have a clearly visible period, as shown in Figs. 4(e) and 4(h).

It is important to understand, however, that not all the moiré-free superpositions are singular (and hence unstable). For example, the superposition of two identical gratings at an angle of 90° is indeed moiré free; however, it is not a singular state, but rather a *stable moiré-free state*: as shown in the first row of Fig. 4, no moiré becomes visible in this superposition even when a small deviation occurs in the angle or in the frequency of any of the layers. The corresponding situation in the spectral domain is clearly illustrated in Fig. 4(c), which shows the vector diagram of the superposition of Fig. 4(b).

Formally, we say that a singular moiré state occurs whenever a $(k_1,\ldots,k_m)$ impulse [other than (0,...,0)] in the spectrum convolution falls exactly on the spectrum origin, i.e., when the frequency vectors of the m superposed gratings, $\mathbf{f}_1,\ldots,\mathbf{f}_m$, are such that $\Sigma_{i=1}^{m} k_i\mathbf{f}_i=\mathbf{0}$. This implies, of course, that all the impulses of the $(k_1,\ldots,k_m)$-moiré comb

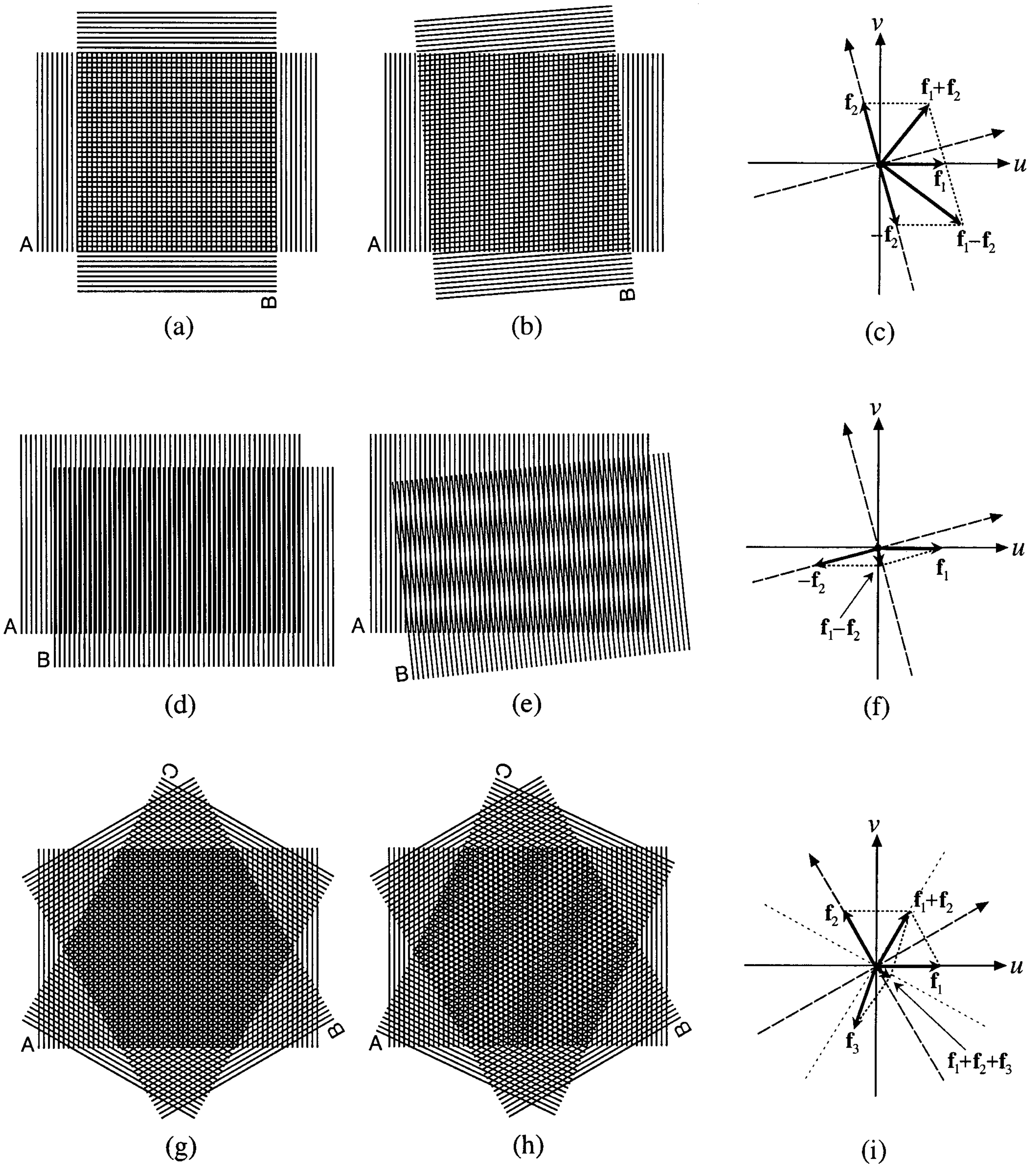

Fig. 4 Examples of stable and unstable (=singular) moiré-free states. First row: (a) the superposition of two identical gratings at an angle difference of 90° gives a stable moiré-free state; small angle or frequency deviations, as in (b), do not cause the appearance of any visible moiré. The spectral interpretation of (b) is shown in the vector diagram (c). Second row: (d) the superposition of two identical gratings at an angle difference of 0° gives a singular (unstable) moiré-free state. (e) A small angle or frequency deviation in any of the layers causes the reappearance of the moiré with a very significant visible period. The spectral interpretation of (e) is shown in the vector diagram (f); compare to Fig. 3(f) which also shows impulses of higher orders. Third row: (g) the superposition of three identical gratings with angle differences of 120° gives an unstable (singular) moiré-free state; again, any small angle or frequency deviation may cause the reappearance of a very significant moiré, as shown in (h) and in its vector diagram, (i).